ON GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS POSSESSING DISCONTINUOUS SOLUTIONS

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PMM Vol.22,No.1, 1958, pp. 27-45.

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(Received 12 May 1957)

The following generalized differential equation is considered:

$$\frac{dx}{dt} = DF(x, t) \qquad (0.1)$$

This equation can be reduced to the ordinary equation $dx/dt = f(x,t)$, if the partial derivative $\partial F / \partial t = f(x,t)$ is continuous. The existence of a solution of the equation (0.1) will be established and it will be shown that the solution is a continuous function of the parameter if the function $F(x,t)$ is continuous in x and of bounded variation in t when x is fixed.

In particular, it is found that the solution of the equation $dx/dt = f(x,t) + d(t)$ is near to some (completely determined) discontinuous function, if $f(x,t)$ is continuous, and the function $d(t)$ is near the Dirac function, i.e.

$$d(t) \geqslant 0, \qquad d(t) = 0 \quad \text{for} \quad |t| \geqslant \delta > 0, \qquad \int_{-\infty}^{\infty} d(t)\, dt = 1$$

We recall certain definitions and results given in an earlier paper [1] which will be used in the sequel

Let $\delta(r)$ be a positive function defined for $r_* \leqslant r \leqslant r^*$. Let the real-valued function $U(r, t)$ be defined for $r_* \leqslant r \leqslant r^*$, $r - \delta(r) \leqslant t \leqslant r + \delta(r)$. The real function $M(r)$, $r_* \leqslant r \leqslant r^*$ is called an upper function for the function U if there exists a positive function $\delta'(r) \leqslant \delta(r)$, $r_* \leqslant r \leqslant r^*$ such that

$$(\tau - \tau_0)(M(\tau) - M(\tau_0)) \geqslant (\tau - \tau_0)(U(\tau_0, \tau) - U(\tau_0, \tau_0))$$

for

$$\tau_0 - \delta'(\tau_0) \leqslant \tau \leqslant \tau_0 + \delta'(\tau_0) \qquad (0.2)$$

A function $m(r)$ will be called a lower function for U if the function $-m(r)$ is an upper function for the function $-U$. For an

Reprint Order No. PMM 3.

arbitrary upper function $M(\tau)$ for U, and for an arbitrary lower function $m(\tau)$ for U it is always true that

$$M(\tau^*) - M(\tau_*) \geqslant m(\tau^*) - m(\tau_*) \tag{0.3}$$

This inequality justifies the following definitions.

If $U(\tau, t)$ is such that there exists for it an upper function $M(\tau)$ and a lower function $m(\tau)$, and if

$$\inf_M (M(\tau^*) - M(\tau_*)) = \sup_m (m(\tau^*) - m(\tau_*)) \tag{0.4}$$

where $M(\tau)$ is the set of all upper, and $m(\tau)$ the set of all lower functions for U, then U is said to be integrable (according to Perrone in the generalized sense), and the number $I = \inf_M [M(\tau^*) - M(\tau_*)]$ is called the generalized integral (according to Perrone) of DU with the limits of integration from τ_* to τ^*:

$$I = \int_{\tau_*}^{\tau^*} DU(\tau, t) \tag{0.5}$$

For the P-integral thus defined, certain basic theorems can be proved, for example,

$$\int_{\tau_*}^{\tau^*} DU(\tau, t) = \int_{\tau_*}^{\sigma} DU(\tau, t) + \int_{\sigma}^{\tau^*} DU(\tau, t) \tag{0.6}$$

if $\tau_* \leqslant \sigma \leqslant \tau^*$ and if the integrals appearing on the right-hand side or the integral on the left-hand side of the equation are defined.

If $U(\tau, t) = f(\tau)t$, then the integral (0.5) exists only when

$$\int_{\tau_*}^{\tau^*} f(\tau)\, d\tau$$

exists in the sense of Perrone, and in this case the two integrals are equal.

If the function $U(\tau, t) = (U_1(\tau, t), \ldots, U_n(\tau, t))$ takes on values in a Euclidean space E_n, then $U(\tau, t)$ is said to be integrable if each of the functions $U_1(\tau, t), \ldots, U_n(\tau, t)$ is integrable. In this case

$$\int_{\tau_*}^{\tau^*} DU(\tau, t) = \left(\int_{\tau_*}^{\tau^*} DU_1(\tau, t), \ldots, \int_{\tau_*}^{\tau^*} DU_n(\tau, t)\right)$$

We next give the definition of the generalized differential equation (0.1). Let G be an open subset of the Euclidean space E_{n+1}, and let the function $F(x, t)$ be defined for $(x, t) \in G$, $x = (x_1, \ldots, x_n)$, and take on values in the Euclidean space E_n. Let the function $x(\tau)$ be defined in $\tau_1 \leq \tau \leq \tau_2$, take on values in E_n, and let $(x(\tau), \tau) \in G$ for $\tau_1 \leq \tau \leq \tau_2$.

The function $x(\tau)$ is said to be a solution of the generalized differential equation (0.1) if

$$x(\tau_4) = x(\tau_3) + \int_{\tau_3}^{\tau_4} DF(x(\tau), t) \text{ for } \tau_3, \tau_4 \in \langle \tau_1, \tau_2 \rangle \tag{0.7}$$

Remark 0.1. The given definition of a generalized differential equation is a particular case of the definition introduced in an earlier article [1], where it was assumed that the function F was defined on some subset of the space E_{n+2} and depended on the variables x, τ, t.

It is proved that all solutions of the generalized equation (0.1) are also solutions of the classical equation

$$dx / dt = f(x, t) \tag{0.8}$$

if the derivative $\partial F(x,t)/\partial t = f(x,t)$ is continuous; and, conversely, every solution of (0.8) is also a solution of (0.1).

The existence of the integral (0.5) is proved below; the existence theorem of a solution for equation (0.1) is proved in Section 2. The proof is similar to the proof of the corresponding theorem under Carathéodory's conditions.

The continous dependence of the solution on the parameter is considered in Section 3. In Section 4, the uniqueness theorem is proved under the assumption that $\omega(\eta) = c\,\eta \; (c > 0)$.

1. Existence of the integral (0.5)

We introduce the following notation:

$h_1(t)$, $h_2(t)$ are functions, defined for $t \in \langle T_1, T_2 \rangle$, non-decreasing and continuous from the left.

$\omega(\eta)$ is a function defined for $0 \leqslant \eta \leqslant \eta_0$, $\eta_0 > 0$, continuous, increasing, $\omega(\eta) \not\geqslant c\eta \quad (c > 0)$, $\omega(0) = 0$.

G is an open subset of the n-dimensional Euclidean space $E_n(T_1, T_2)$, $(T_2 > T_1)$. $F = F(G, \omega, h_1)$ is the class of functions $F(x,t)$ satisfying the following conditions: the function $F(x,t)$ is defined for $(x,t) \in G$ and takes on values in E_n:

$$\| F(x, t_1) - F(x, t_2) \| \leqslant | h_1(t_2) - h_1(t_1) |, \quad \text{if } (x, t_1), (x, t_2) \in G$$

$$\| F(x_2, t_2) - F(x_2, t_1) - F(x_1, t_2) + F(x_1, t_1) \| \leqslant$$
$$\leqslant \omega(\| x_2 - x_1 \|) \, | h_1(t_2) - h_1(t_1) |$$
$$\text{if } (x_1, t_1), (x_1, t_2), (x_2, t_1) \; (x_2, t_2) \in G, \; \| x_2 - x_1 \| \leqslant \eta_0$$

$u(\tau)$ is a function defined for $\tau \in (\sigma_1, \sigma_2)$, $(\sigma_2 > \sigma_1)$, $u(\tau) \in E_n$.
$(u(\tau), \tau) \in G$, $\| u(\tau_2) - u(\tau_2) \| \leqslant | h_2(\tau_2) - h_2(\tau_2) |$, for $\tau_1, \tau_2 \in (\sigma_1, \sigma_2)$.

$N(\eta)$ is the set of all points $t \in (T_1, T_2)$, for which

$$\text{either} \quad \omega\,(h_2(t+) - h_2(t)) \geqslant \eta,$$
$$\text{or} \quad h_2(t+) - h_2(t) > \eta_0,$$

where η is an arbitrary positive number and

$$h_2(t+) = \lim h_2(\tau) \quad \text{as} \quad \tau \to t, \quad \tau > t.$$

(The cardinal number of the set $N(\eta)$ is obviously finite).

$A(\eta, \tau_1, \tau_2, h_2)$ is the set whose elements are finite sequences of numbers

$$\{\alpha_0, \tau_1, a_1, \tau_2, \ldots, \tau_s, \alpha_s\} = A$$

if the following conditions are fulfilled:

(1) $\sigma_1 = \alpha_0 < \alpha_1 < \ldots < \alpha_3 = \sigma_2$;

(2) $\alpha_0 \leq \tau_1 \leq \alpha_1 \leq \tau_2 \leq \ldots \leqslant \tau_s \leqslant \alpha_s$;

(3) if $t \in N(\eta) \cap \langle \sigma_1, \sigma_2 \rangle$, then there exists an index j such that $t = \tau_j$ and $\tau_j < \alpha_j$; in the case that $t = \sigma_2 \epsilon N(\eta)$, it is only required that $t = \tau_j$ for the appropriate j;

(4) if $\tau_j \bar{\epsilon} N(\eta)$, then

$$h_2(\alpha_j) - h_2(\alpha_{j-1}) < \eta_0, \quad \omega\,(h_2(\alpha_j) - h_2(\alpha_{j-1})) < \eta,$$

if $\tau_j \epsilon N(\eta)$, then

$$h_2(\alpha_j) - h_2(\tau_j+) < \eta_0, \quad \omega\,(h_2(\alpha_j) - h_2(\tau_j+)) < \eta \qquad (\text{for } \tau_j < \sigma_2)$$
$$h_2(\tau_j) - h_2(\alpha_{j-1}) < \eta_0, \quad \omega\,(h_2(\delta_j) - h_2(\alpha_{j-1})) < \eta$$

The sequence A will be called a subdivision of the interval $\langle \sigma_1, \sigma_2 \rangle$.

For the formulation of the theorem on the existence of the integral we need the following lemma.

Lemma 1.1. The set $A(\eta, \sigma_1, \sigma_2, h_2)$ is non-empty.

For the proof of this lemma we select for every $\tau \epsilon \langle \sigma_1, \sigma_2, \rangle$ a positive number $\delta(\tau)$ satisfying the following conditions:

if $\tau \bar{\epsilon} N(\eta)$, then

$$\delta(\tau) < \eta_0, \qquad \omega\,(h_2(\zeta) - h_2(\tau)) < \eta, \qquad \omega\,(h_2(\tau) - h_2(\zeta') < \eta)$$
$$\tau \in N(\eta),$$

if $\tau \epsilon N(\eta)$, then

$$\delta(\tau) < \eta_0, \qquad \omega\,(h_2(\zeta') - h_2(\tau+)) < \eta, \qquad \omega\,(h_2(\tau) - h_2(\zeta')) < \eta$$

where $\zeta = \zeta(\tau) = \min(\sigma_2, \tau + \delta(\tau))$, $\quad \zeta' = \zeta'(\tau) = \max(\sigma_1, \tau - \delta(\tau))$

Obviously, the interval (ζ', ζ) does not contain a point of $N(\eta)$ if $\tau \bar{\epsilon} N(\eta)$, and the intersection of the interval (ζ', ζ) with the set $N(\eta)$ contains only the point τ if $\tau \epsilon N(\eta)$.

By decreasing $\delta(\tau)$, one can establish that the above assertion is true also for the closed interval $<\zeta'(r),\ \zeta(r)>$.

Let the intervals $<\zeta'(r_1),\ \zeta(r_1)>,\ \ldots,\ <\zeta'(r_s),\ \zeta(r_s)>$ cover the interval $<\sigma_1,\ \sigma_2,\ >$, $r_1 < r_2 < \ldots < r_2$, and suppose that the interval $<\sigma_1,\ \sigma_2>$ is no longer covered if one of the intervals $<\zeta'(r_j),\zeta(r_j)>$ is omitted. This property implies that $\zeta'(r_j) < \zeta'(r_{j+1})$, $\zeta(r_j) < \zeta(r_{j+1})$, $\zeta(r_{j-1}) < \zeta'(r_{j+1})$.

From these inequalities it follows that there exist numbers $\alpha_0,,..,\alpha$ such that the subdivision $\{\alpha_0,\ r_1,\ \ldots,\ r_s,\ \alpha_s\}$ satisfies all of our requirements. In particular, if $t \in <\sigma_1,\ \sigma_2> \cap N(\eta)$, then there exists an index j such that $t = r_j$, for $t \in <\zeta'(r),\ \zeta(r)>$ only if $t = r$. Greater details of these considerations are given in reference [1], § 1, in the proof of Lemma 1.1.

Theorem 1.1. The integral

$$\int_{\sigma_1}^{\sigma_2} DF(u(\tau),\ t) \tag{1.1}$$

exists if

$\{\alpha_0,\ \tau_1,\ \alpha_1,\ \ldots,\ \tau_s,\ \alpha_s\} \in A(\eta,\ \sigma_1,\ \sigma_2,\ h_2)$, then

$$\left\|\int_{\sigma_1}^{\sigma_2} DF(u(\tau),\ t) - \sum_{j=1}^{s} R_j\right\| \leqslant \eta\sqrt{n}\,[h_1(\sigma_2) + h_2(\sigma_2) - h_1(\sigma_1) - h_2(\sigma_1)]$$

where $R_j = F(u(\tau_j),\ \alpha_j) - F(u(\tau_j),\ \alpha_{j-1})$, if $\tau_j \bar{\in} N(\eta)$ or $\tau_j = \sigma_2$

$$R_j = F(u(\tau_j +),\ \alpha_j) - F(u(\tau_j +),\ \tau_j +) + F(u(\tau_j),\ \tau_j +) -$$
$$- F(u(\tau_j),\ \alpha_{j-1}),\ \text{ if } \tau_j \in N(\eta),\ \tau_j < \sigma_2$$

Corollary 1.1.

$$\left\|\int_{\sigma_3}^{\sigma_4} DF(u(\tau),\ t)\right\| \leqslant |h_1(\sigma_4) - h_1(\sigma_3)| \qquad \text{for } \sigma_3,\ \sigma_4 \in \langle\sigma_1,\ \sigma_2\rangle$$

Proof: Let $U(r,t)$ be an arbitrary component of the vector function $F(u(r),t)$, and let $\eta > 0$. If $r_0 \in <\sigma_1,\ \sigma_2>$ $r_0 \bar{\in} N(\eta)$ let us set

$$M(\tau_0,\ \tau) = U(\tau_0,\ \tau) + \eta(h_1(\tau) + h_2(\tau))$$
$$m(\tau_0,\ \tau) = U(\tau_0,\ \tau) - \eta(h_1(\tau) + h_2(\tau)) \qquad \tau \in \langle\tau_1,\ \tau_2\rangle$$

If $r_0 \in <\sigma_1,\ \sigma_2> \cap N(\eta)$, let

$$M(\tau_0,\ \tau) = U(\tau_0,\ \tau) + \eta(h_1(\tau) + h_2(\tau)) \quad \text{for } \sigma_1 \leqslant \tau \leqslant \tau_0$$
$$M(\tau_0,\ \tau) = U(\tau_0,\ \tau_0 +) + U(\tau_0 +,\ \tau) - U(\tau_0 +,\ \tau_0 +) + \eta(h(\tau) + h_2(\tau)),$$
$$\text{for } \tau_0 < \tau \leqslant \sigma_2$$

$$m(\tau_0,\ \tau) = U(\tau_0,\ \tau) - \eta(h_1(\tau) + h_2(\tau)) \quad \text{for } \sigma_1 \leqslant \tau \leqslant \tau_0$$
$$m(\tau_0,\ \tau) = U(\tau_0,\ \tau_0 +) + U(\tau_0 +,\ \tau) - U(\tau_0 +,\ \tau_0 +) - \eta(h_1(\tau) + h_2(\tau))$$
$$\text{for } \tau_0 < \tau \leqslant \sigma_2$$

We define the function $M(\tau)$, for $\tau \in <\sigma_1, \sigma_2>$, the following way:

$$M(\sigma_1) = 0$$

$$M(\tau) = \sum_{j=1}^{i-1} [M(\tau_j, \alpha_j) - M(\tau_j, \alpha_{j-1})] + M(\tau_i, \tau) - M(\tau_i, \alpha_{i-1})$$

$$\text{if } \alpha_{i-1} < \tau \leqslant \alpha_i$$

We prove that $M(\tau)$ is an upper function for $U(\tau, t)$. The inequality

$$(\tau - \tau_0)[M(\tau) - M(\tau_0)] \geqslant (\tau - \tau_0)[U(\tau_0, \tau) - U(\tau_0, \tau_0)]$$

is obviously satisfied if $\tau_0 \in <\sigma_1, \sigma_2> \cap N(\eta)$, $\tau_0 \leqslant \tau \leqslant \sigma_2$ and if τ is near enough to τ_0.

Let $\tau, \tau_0 \in <\alpha_{i-1}, \alpha_i>$ if $\tau_i \bar{\in} N(\eta)$. If $\tau_i \notin N(\eta)$, let $\tau, \tau_0 \in <\alpha_{i-1}, \tau_i>$. We obtain

$$(\tau - \tau_0)[U(\tau_0, \tau) - U(\tau_0, \tau_0)] \leqslant (\tau - \tau_0)[U(\tau_i, \tau) - U(\tau_i, \tau_0)] +$$
$$+ |\tau - \tau_0| \, |U(\tau_0, \tau) - U(\tau_0, \tau_0) - U(\tau_i, \tau) + U(\tau_i, \tau_0)| \leqslant$$
$$\leqslant (\tau - \tau_0)[U(\tau_i, \tau) - U(\tau_i, \tau_0)] + |\tau - \tau_0| \, \omega(\|u(\tau_i) - u(\tau_0)\|) \, |h_1(\tau) - h_1(\tau_0)| \leqslant$$
$$\leqslant (\tau - \tau_0)[U(\tau_i, \tau) - U(\tau_i, \tau_0) + \eta \, (h_1(\tau) - h_1(\tau_0))] = (\tau - \tau_0)(M(\tau) - M(\tau_0)$$

Here we have used the following inequalities

$$\|u(\tau_i) - u(\tau_0)\| \leqslant h_2(\alpha_i) - h_2(\alpha_{i-1}) \leqslant \eta_0$$
$$\omega(\|u(\tau_i) - u(\tau_0)\|) \leqslant \omega(h_2(\alpha_i) - h_2(\alpha_{i-1})) < \eta$$
$$|U(\tau_0, \tau) - U(\tau_0, \tau_0) - U(\tau_i, \tau) + U(\tau_i, \tau_0)| \leqslant$$
$$\leqslant \|F(u(\tau_0), \tau) - F(u(\tau_0), \tau_0) - F(u(\tau_i), \tau) + F(u(\tau_i), \tau_0)\| \leqslant$$
$$\leqslant \omega(\|u(\tau_0) - u(\tau_i)\|) \, |h_1(\tau) - h_1(\tau_0)|$$

The case that remains to be considered is when $\tau_i \in N(\eta)$, $\tau_0, \tau \in <_i, \alpha_i>$. In this case we obtain in a manner analogous to the one used above

$$(\tau - \tau_0)[U(\tau_0, \tau) - U(\tau_0, \tau_0)] \leqslant (\tau - \tau_0)[U(\tau_i +, \tau) - U(\tau_i +, \tau_0)] +$$
$$+ |\tau - \tau_0| \, |U(\tau_0, \tau) - U(\tau_0, \tau_0) - U(\tau_i +, \tau) + U(\tau_i +, \tau_0)| \leqslant$$
$$\leqslant (\tau - \tau_0)[U(\tau_i +, \tau) - U(\tau_i +, \tau_0)] + |\tau - \tau_0| \, \eta \, |h_1(\tau) - h_1(\tau_0)| =$$
$$= (\tau - \tau_0)[U(\tau_i +, \tau) - U(\tau_i +, \tau_0) + \eta(h_1(\tau) - h_1(\tau_0))] =$$
$$= (\tau - \tau_0)[M(\tau) - M(\tau_0)]$$

Here we have used the inequalities

$$|U(\tau_0, \tau) - U(\tau_0, \tau_0) - U(\tau_i +, \tau) + U(\tau_i +, \tau_0)| \leqslant$$
$$\leqslant \|F(u(\tau_0), \tau) - F(u(\tau_0), \tau_0) - F(u(\tau_i +), \tau) + F(u(\tau_i +), \tau_0))\| \leqslant$$
$$\leqslant \omega(\|u(\tau_0) - u(\tau_i +)\|) \, |h_1(\tau) - h_1(\tau_0)| \leqslant \omega(h_2(\alpha_i) - h_2(\tau_i +)) \, |h_1(\tau) - h_1(\tau_0)| \leqslant$$
$$\leqslant \eta \, |h_1(\tau) - h_1(\tau_0)|$$

Thus we have proved that for every $\tau_0 \in <\sigma_1, \sigma_2>$ there is satisfied the inequality (0.2) if only $\tau \in <\sigma_1, \sigma_2>$ is sufficiently near to τ_0. This means that $M(\tau)$ is an upper function for $U(\tau, t)$. In an analogous manner we can construct a lower function $m(t)$. From the inequality

$$M(\sigma_2)-M(\sigma_1)-(m(\sigma_2)-m(\sigma_1))\leqslant 2\eta\,(h_1(\sigma_2)+h_2(\sigma_2)-h_1(\sigma_1)-h_2(\sigma_1))$$

it follows that the integral (0.5) exists, for the functions $M(r)$ and $m(r)$ can be constructed for an arbitrary positive η (and for the corresponding subdivision). Hence, the integral (1.1) exists. Furthermore, in consequence of the inequality

$$M(\sigma_4)-M(\sigma_3)\geqslant\int_{\sigma_3}^{\sigma_4} DU(\tau,t)\geqslant m(\sigma_4)-m(\sigma_3)$$

and from the representation of the functions $M(r)$ and $m(r)$ it follows that

$$\left|\int_{\sigma_3}^{\sigma_4} DU(\tau,t)-\sum_{j=l}^{m} P_j\right|\leqslant\eta\,(h_1(\sigma_4)+h_2(\sigma_4)-h_1(\sigma_3)-h_2(\sigma_3))$$

where P_j is the appropriate component of the vector R_j. This yields the inequality of Theorem 1.1.

2. Existence theorem for the solution of equation (0.1)

Let $F(x,t)$ be a given function of the class $F(G,\omega,h_1)$. We denote by G_F the subset of the set G which consists of those points (x,t) for which $(x+F(x,t+)-F(x,t),t)\in G$. Obviously, if for a fixed t, $h_1(t+)=h_1(t)$ and $(x,t)\in G$, then $(x,t)\in G_F$. It is easily verified that G_F is an open set.

Theorem 2.1. Let $(x_0,\ t_0)\in G$ and let us select a number $\xi>0$ which has the following properties: if $t_0\leqslant t_0+\xi$ and $\|\,x-[x_0+F(x_0,t_0+)-F(x_0,t_0+)]\,\|<h_1(t)-h_1(t_0+)$, then $(x,t)\in G$. Under these conditions there exists a function $x(r)$, defined for $t_0\leqslant r\leqslant t_0+\xi$ which is of bounded variation, is continuous from the left, and satisfies the equation

$$\frac{dx}{d\tau}=DF(x,t)\tag{2.1}$$

with the initial condition $x(t_0)=x_0$.

Remark 2.1. In consequence of (1.1) we find that every solution $u(r)$ of equation (2.1) which is of bounded variation and continuous from the left satisfies the inequality

$$\|\,u(\tau_2)-u(\tau_1)\,\|\leqslant|\,h_1(\tau_2)-h_1(\tau_1)\,|\tag{2.2}$$

for arbitrary r_1, r_2 from the interval on which $u(r)$ is defined. In the article [1], Section 1, there is given another equivalent definition of the integral (0.5) by means of partial sums and a certain limiting process. From that definition it is easily deduced that the inequality (2.2) is valid for any function $u(r)$ which is a solution of equation (2.1), without the assumption that the function $u(r)$ be of bounded variation and continuous from the left. This can be established on the basis of the fact that an inequality analogous to (2.2) can in this case be established for any one of the partial sums which enter into the earlier definition of the integral (0.5).

Remark 2.2. Any solution $x(\tau)$ which is defined on some interval with t_0 as its left endpoint, can be extended for increasing τ (just as in the classical case) as long as the point $(x(\tau),\tau)$ lies in the set G_F. But the solutions cannot always be extended for decreasing τ. Let us consider the most simple case. Suppose $x \in E_1$ and $F(x,t) = x$ for $t \leqslant 0$; $F(x,t) = 0$ for $t > 0$. In terms of the concepts we have introduced, the following theorem is true: if $\tau_* \leqslant \tau_0 < \tau^*$, and the integral (0.5) exists for the interval $<\tau_*, \tau^*>$, then the integral (0.5) exists for the interval $<\tau_*, \tau_0>$ and if the lim $U(\tau_0, t)$ as $t \to \tau_0 +$ exists, then

$$\lim_{\tau_1 \to \tau_0 +} \int_{\tau_*}^{\tau_1} DU(\tau, t) = \int_{\tau_*}^{\tau_0} DU(\tau, t) + \lim_{t \to \tau_0 +} U(\tau_0, t) - U(\tau_0, \tau_0) \tag{2.2*}$$

(see [1), theorems 1.3.3 and 1.3.6). In consequence of this it follows that lim $x(\tau) = 0$ as $\tau \to 0+$, where $x(t)$ is an arbitrary solution defined in some neighborhood of the number $t = 0$. For the given example, the solution $x_1(\tau)$, defined by the equation $x_1(\tau) = 1$ for $\tau > 0$, cannot be extended to $\tau \leqslant 0$.

Proof. We define a sequence of functions $x_i(\tau)$:

$$x_i(\tau) = x_0 \qquad \text{for } t_0 - \xi/i \leqslant \tau \leqslant t_0$$

$$x_i(\tau_i) = x_0 + \int_{t_0}^{\tau_1} DF(x_i(\tau - \xi/i), t) \qquad \text{for } t_0 < \tau_1 \leqslant t_0 + \xi$$

Making use of Theorem 1.1 and Corollary 1.1 one can verify that the functions $x_i(\tau)$ are uniquely defined for $t_0 - \xi/i \leqslant \tau \leqslant t_0 + \xi$ for all integers $i > i_0$, and that

$$\begin{gathered} \| x_i(\tau_2) - x_i(\tau_1) \| \leqslant | h_1(\tau_2) - h_1(\tau_1) | \quad \text{for} \quad \tau_1, \tau_2 \in \langle t_0 - \xi/i, t_0 + \xi \rangle \\ x_i(t_0 +) = x_0 + F(x_0, t_0 +) - F(x_0, t_0) \qquad (i \geqslant i_0) \end{gathered} \tag{2.3}$$

In consequence of the inequality (2.3), one can select from the sequence $x_i(\tau)$ a subsequence which converges uniformly in the interval $< t_0, t_0 + \xi >$. We, therefore, suppose that $x_i(\tau) \to x(\tau)$ uniformly for $\tau \in < t_0, t_0 + \xi >$, $i \to \infty$.

Obviously, $\| x_2(\tau_2) - x(\tau_1) \| \leqslant | h_1(\tau_2) - h_1(\tau_1) |$ and $x_i(\tau - \xi/i) \to x(\tau)$ (but not uniformly), for the function $h_1(\tau)$ is continuous from the left.

Let us evaluate the integral (1.1), where $t < \sigma_1 < \sigma_2 < t_1 + \eta$.

Let η be any positive number less than η_0. Let us take a fixed subdivision $\{\alpha_0, \tau_1, \alpha_1, \ldots, \tau_s, \alpha_s\} \in A(\eta, \sigma_1, \sigma_2, h_1)$.

By Theorem 1.1, the integral (1.1) differs from the partial sum

$$\sum_{\substack{j \\ \tau_j \tilde{\in} N(\eta),\ \text{or } j=s}} [F(x(\tau_j), \alpha_j) - F(x(\tau_j), \alpha_{j-1})] +$$
$$+ \sum_{\substack{j \\ \tau_j \in N(\eta),\ j<s}} [F(x(\tau_j+), \alpha_{j+1}) - F(x(\tau_j+), \tau_j+) + \tag{2.4}$$
$$+ F(x(\tau_j), \tau_j+) - F(x(\tau_j), \alpha_{j-1}))]$$

by a term whose norm is not greater than $K\eta$, where K is independent of η.

Let us set $\tau'_{j,i} = \tau_j$ if $\tau_j \tilde{\in} N(\eta)$, or $j = s$, $\tau'_{j,i} = \tau_j + \xi/i$ if $j < s$ and $\tau_j \in N(\eta)$. Let $g_i(t) = h_1(t_0)$ if $t_0 \leqslant t \leqslant t_0 + \xi/i$, and let $g_i(t) = h_1(t - \xi/i)$ for $t_0 + \xi/i < t_0 + \xi$.

Thus $\{\alpha, \tau'_{1,i}, \alpha_1, \tau'_{2,i}, \ldots, \tau'_{s,i}, \alpha_s\} \in A(\eta, \sigma_1, \sigma_2, h_2)$ for sufficiently large i, and, hence by Theorem 1.1, the integral

$$\int_{\sigma_1}^{\sigma_2} DF(x_i(\tau - \xi/i), t)$$

can be expressed as the partial sum

$$\sum_{\substack{j \\ \tau_j \tilde{\in} N(\eta),\ \text{or } j=s}} [F(x_i(\tau_j - \xi/i), \alpha_j) - F(x_i(\tau_j - \xi/i), \alpha_{j-1})] +$$
$$+ \sum_{\substack{j \\ \tau_j \in N(\eta),\ j<s}} [F(x_i(\tau_j+), \alpha_{j+1}) - F(x_i(\tau_j+), \tau_j+) + \tag{2.5}$$
$$+ F(x_i(\tau_j), \tau_j+) - F(x(\tau)_j, \alpha_{j-1})]$$

to within an error whose norm is not greater than $K\eta$, where K is independent of i and η for sufficiently large i. Since $x_i(\tau) \to x(\tau)$, $x_i(\tau - \xi/i) \to x(\tau)$ and $F(x,t) \in \mathbf{F}$, it follows that the sum (2.4) and (2.5) are arbitrarily close to each other for i sufficiently large. Therefore,

$$\int_{\sigma_1}^{\sigma_2} DF(x_i(\tau - \xi/i), t) \to \int_{\sigma_1}^{\sigma_2} DF(x(\tau), t) \quad \text{for} \quad i \to \infty$$

and $x(\tau)$ is a solution of the equation (2.1). This completes the proof of the theorem.

3. The continuous dependence of the solution on the parameter

(The basic result is Theorem 3.1. The later theorems are obtained as special cases of it.)

Let $F(x,t)$ be a function of class $\mathbf{F}(G, \omega, h)$ and $F_k(x,t)$ be a function of class $\mathbf{F}(G, \omega, h_k)$, where $k = 3, 4, 5, \ldots$, $h(t)$ and $h_k(t)$ are increasing functions in $t \in \langle 0, T\rangle$, $(G \subset E_n\langle 0,T\rangle)$. Furthermore, $h(t)$ and $h_k(t)$ are continuous from the left. Let H be the set of points of discontinuities of the function $h(t)$, G_1 be some open subset of G_F which contains all points $(x,t) \in G$ for which $t \in H$.

We shall say that the sequence $F_k(x,t)$ converges unconditionally to the function $F(x,t)$ in G_1 , and write $F(x,t) \twoheadrightarrow F(x,t)$, if the following properties are exhibited

(I) $\lim\limits_{k\to\infty} \sup (h_k(t_2) - h_k(t_1)) \leqslant h(t_2) - h(t_1)$ for $t_1, t_2 \in H$; $t_1 < t_2$

(II) $F_k(x,t) \to F(x,t)$ for $k \to \infty$ $(x,t) \in G_1$, $t \in H$

(III) if $(x_0, t_0) \in G_1$, $t_0 \notin H$, then for every $\epsilon > 0$ there exist numbers δ_1, δ_2 having the following properties: for arbitrary numbers t_1, $t_2 \in H$, $t_0 - \delta_1 < t_1 < t_0 < t_2 < t_0 + \delta_1$, there exists a $K > 0$ such that

(a) if $\|y - x_0\| < \delta_2$ and $k > K$ then there exists a function $x_k(\tau)$, defined in $t_1 \leqslant \tau \leqslant t_2$, which is a solution of the equation

$$\frac{dx}{d\tau} = DF_k(x,t), \qquad x_k(t_1) = y$$

(b) every function $x_k(\tau)$ satisfying (a), satisfies also the inequality $\|x_k(t_2) - x_k(t_1) - F(x_0, t_0+) + F(x_0, t_0)\| < \epsilon$, and there exists a positive number μ, independent of the solution $x_k(\tau)$, of the subscript k, and of the point y, such that the distance of an arbitrary point $(x_k(\tau), \tau)$ $(t_1 \leqslant \tau \leqslant t_2,\ k > K)$ from the boundary of the set G_1 is greater than μ.

Theorem 3.1. Let $F_k(x,t) \rightrightarrows F(x,t)$ in G_1, let $x(t)$ a solution of the equation

$$\frac{dx}{d\tau} = DF(x,t) \tag{3.1}$$

be uniquely defined in the interval $\langle \sigma_1, \sigma_2 \rangle$ by its initial condition, $\sigma_1, \sigma_2 \in H$, $(x(\tau), \tau) \in G_1$ for $\tau \in \langle \sigma_1, \sigma_2 \rangle$. Let $y_k \in E_n$, $y_k \to x(\sigma_1)$.

Then:

(c) for all large enough k there exists a solution $x_k(\tau)$ of the equation

$$\frac{dx}{d\tau} = DF_k(x,t) \tag{3.2}$$

defined for $\tau \in \langle \sigma_1, \sigma_2 \rangle$, $x_k(\sigma_1) = y_k$, and the distance of an arbitrary point $(x_k(\tau), \tau)$ from the boundary of the set G_1 is greater than some positive number μ;

(d) if $x_k(\tau)$ is a sequence of functions satisfying condition (c), then

$$x_k(\tau) \to x(\tau) \quad \text{for } k \to \infty,\ \tau \in \langle\sigma_1, \sigma_2\rangle \cap H$$

Remark: We shall say that the solution $x(\tau)$ is uniquely defined by its initial condition if each solution $u(\tau)$ of the equation (3.1) defined in some interval $\langle\sigma_1, \sigma_2'\rangle \subset \langle\sigma_1, \sigma_2\rangle$, $u(\sigma_1) = x(\sigma_1)$, coincides with $x(\tau)$ on $\langle\sigma_1, \sigma_2'\rangle$.

First we prove the following lemma.

Lemma 3.1. If the condition (c) of Theorem 3.1 is satisfied on some interval $\langle\sigma_1, \sigma_2'\rangle \subset \langle\sigma_1, \sigma_2\rangle$, $\sigma_2' \in H$, then the condition (d) is also satisfied on the same interval $\langle\sigma_1, \sigma_2'\rangle$.

Proof: Let the sequence $x_k(\tau)$ satisfy condition (c) of Theorem 3.1 on the interval $\langle\sigma_1, \sigma_2'\rangle$. Then there exists a subsequence $u_k(\tau)$ which converges for each $\tau \in \langle\sigma_1, \sigma_2'\rangle \cap H$. (This follows from the fact that the functions $u_k(\tau)$ satisfy the inequality $\|u_k(\lambda_2) - u_k(\lambda_1)\| < |h_k(\lambda_2) - h_k(\lambda_1)|$, $\lambda_1, \lambda_2 \in \langle\sigma_1, \sigma_2'\rangle$, in consequence of 2.1). Suppose

$$u(\tau) = \lim_{k \to \infty} u_k(\tau), \qquad \tau \in \langle\sigma_1, \sigma_2'\rangle \cap H$$

Then we have

$$\|u(\lambda_2) - u(\lambda_1)\| \leqslant |h(\lambda_2) - h(\lambda_1)|, \quad \lambda_1, \lambda_2 \in \langle\sigma_1, \sigma_2'\rangle \cap H \tag{3.3}$$

Let us extend the definition of the function $u(\tau)$ in such a manner that the function $u(\tau)$ be defined and continuous from the left in the interval $\langle\sigma_1, \sigma_2'\rangle$. Then (3.3) will be valid for $\lambda_1, \lambda_2 \in \langle\sigma_1, \sigma_2'\rangle$. Let σ_3 be an arbitrary number in the interval $\langle\sigma_1, \sigma_2\rangle$, $\sigma_3 > \sigma_1$, $\sigma_3 \in H$ Our immediate aim is to prove that

$$\int_{\sigma_1}^{\sigma_3} DF_k(u_k(\tau), t) \to \int_{\sigma_1}^{\sigma_3} DF(u(\tau), t) \quad \text{for } k \to \infty \tag{3.4}$$

Suppose $\eta > 0$. Let $\tilde{\tau}_1 < \tilde{\tau}_2 \ldots < \tilde{\tau}_r$ be points of the set $\langle\sigma_1, \sigma_3\rangle \cap N(\eta)$. In consequence of our hypothesis it follows that $(u(\tilde{\tau}_j), \tilde{\tau}_j) \in G_i$. For the numbers $\epsilon = \eta/i$ and the points $(u(\tilde{\tau}_j), \tilde{\tau}_j)$ we select numbers $\delta_{1j} > 0$, $\delta_{2j} > 0$ such that the condition (III), on the unconditional convergence, be satisfied; let δ_j be such a positive number that

$$\begin{gathered} h(\tilde{\tau}_j) - h(\tilde{\tau}_j - \delta_j) + h(\tilde{\tau}_j + \delta_j) - h(\tilde{\tau}_j +) < \eta / r \\ \delta_j < \delta_{1j}, \qquad \omega(h(\tilde{\tau}_j) - h(\tilde{\tau}_j - \delta_j)) < \delta_{2j} \end{gathered} \tag{3.5}$$

In a manner similar to the one used in Lemma 1.1 we prove that there exists a subdivision $\{\alpha_0, \tau_1, \ldots, \tau_s, \alpha_s\} \in A(\eta, \sigma_1, \sigma_2\ h)$ which satisfies the following conditions:

(e) let $\tau_{p_1} < \ldots < \tau_{pr}$ be numbers such that $\tau_j \in \langle\sigma_1, \sigma_2\rangle \cap N(\eta)$.

Then $\tau_{p_1} = \bar{\tau}_1, \tau_{p_2} = \bar{\tau}_2, \ldots, \tau_{p_r} = \bar{\tau}_r$, and hence

$$\tau_{p_j} - \delta_j < \alpha_{p_j-1} < \tau_{p_j} < \alpha_{p_j} < \tau_{p_j} + \delta_j$$

(f) if $\tau_j \bar{\epsilon} N(\eta)$, then $\tau_j \epsilon H$;

(g) $\alpha_j \epsilon H$, $j = 0, 1, \ldots, s$.

(For the construction of the subdivision we start with a system of intervals J defined in the following way: the interval $\langle \zeta'(\tau), \zeta(\tau) \rangle$ is one of the intervals J if $\tau \epsilon \langle \sigma_1, \sigma_2' \rangle \cap N(\eta)$ and hence $\tau = \tau_j$, $\tau_j - \delta_j < \zeta'(\tau_j) < \tau_j < \zeta(\tau_j) < \tau_j + \delta_j$, $\omega(h(\zeta(\tau_j)) - h(\tau_j +)) < \eta$, $\omega(h(\tau_j) - h(\zeta'(\tau_j)) < \eta$, or if $\tau \epsilon N(\eta)$, $\tau \epsilon \langle \sigma_1, \sigma_2' \rangle$, and $\zeta(\tau)$, $\zeta'(\tau)$ are numbers satisfying the inequalities $\sigma_1 \leqslant \zeta'(\tau) \leqslant \tau < \zeta(\tau) \leqslant \sigma_2'$, $\omega(h(\zeta(\tau) - h(\zeta'_{\prime})(\tau) < \eta$, $\zeta'(\tau) < \tau$, if $\sigma_1 < \tau$, $\zeta(\tau) > \tau_1$, when $(\tau) < \sigma_2'$.

It is easily seen that the intervals of the system J cover the interval $\langle \sigma_1, \sigma_2' \rangle$ and sc on).

Obviously, $\{\alpha_{p_{j-1}}, \tau_{p_{j-1}} + 1, \ldots, \alpha_{p_j} - 1\}$ $\epsilon\ A(\eta, \alpha_{p_{j-1}}, \alpha_{p_j-1}, h)$, where $j = 1, \ldots, r+1$, $\alpha_{p_0} = \alpha_0$, $\alpha_{p_{r+1}} = \alpha_s$, and from Theorem 2.1 it follows that

$$\left\| \int_{\alpha_{p_{j-1}}}^{\alpha_{p_j-1}} DF(u(\tau), t) - \sum_{i=p_{j-1}+1}^{p_j-1} [F(u(\tau_i), \alpha_i) - F(u(\tau_i), \alpha_{i-1})] \right\| \leqslant \leqslant 2\eta [h(\alpha_{p_j-1}) - h(\alpha_{p_{j-1}})] \tag{3.6}$$

Furthermore, for all sufficiently large k

$$\{\alpha_{p_{j-1}}, \tau_{p_{j-1}+1}, \ldots, \alpha_{p_j-1}\} \in A(\eta, \alpha_{p_{j-1}}, \alpha_{p_j-1}, h_k)$$

and making use of Theorem 2.1 we obtain

$$\left\| \int_{\alpha_{p_{j-1}}}^{\alpha_{p_j-1}} DF_k(u_k(\tau), t) - \sum_{i=p_{j-1}+1}^{p_j-1} [F_k(u_k(\tau_i), \alpha_i) - F_k(u_k(\tau_i), \alpha_{i-1})] \right\| \leqslant \leqslant 3\eta [h(\alpha_{p_j-1}) - h(\alpha_{p_{j-1}})] \tag{3.7}$$

Since τ_i, $\alpha_i \epsilon H$, and $u_k(\tau_i) \to u(\tau_i)$, it follows from the hypothesis on the sequence $F_k(x, t)$ that

$$\sum_{i=p_{j-1}+1}^{p_j-1} [F_k(u_k(\tau_i), \alpha_i) - F_k(u_k(\tau_0), \alpha_{i-1})] \to \to \sum_{i=p_{j-1}+1}^{p_j-1} [F(u(\tau_i), \alpha_i) - F(u(\tau_i), \alpha_{i-1})] \quad \text{as } k \to \infty \tag{3.8}$$

On the basis of (3.6), (3.7) and (3.8) we now obtain

$$\left\| \int_{\alpha_{p_{j-1}}}^{\alpha_{p_j-1}} DF_k(u_k(\tau), t) - \int_{\alpha_{p_{j-1}}}^{\alpha_{p_j-1}} DF(u(\tau), t) \right\| \leqslant 6\eta\, [h(\alpha_{p_j-1}) - h(\alpha_{p_{j-1}})] \quad (3.9)$$

which holds for sufficiently large k, and $j = 1, \ldots, r+1$. Next, let j be one of the integers $1, \ldots, n$. On the basis of Corollary 2.1 and of the last of the inequalities in (3.5)

$$\| u(\alpha_{p_j-1}) - u(\tau_{p_j}) \| \leqslant \omega(h(\tau_{p_j}) - h(\alpha_{p_{j-1}})) < \omega(h(\tilde{\tau}_j) - h(\tilde{\tau}_j - \delta_j)) < \delta_{2j}$$

and, hence, for sufficiently large k,

$$\| u_k(\alpha_{p_j-1}) - u(\tau_{p_j}) \| < \delta_{2j}$$

because $\alpha_{p_j-1} \in H$ and $u_k(\alpha_{p_j-1}) \to u(\alpha_{p_{j-1}})$.

On the basis of the definition of the unconditional convergence, we thus obtain the representation

$$\int_{\alpha_{p_j-1}}^{\alpha_{p_j}} DF_k(u_k(\tau), t) = u_k(\alpha_{p_j}) - u_k(\alpha_{p_{j-1}}) = \quad (3.10)$$
$$= F(u(\tau_{p_j}), \tau_{p_j}+) - F(u(\tau_{p_j}), \tau_{p_j}) + z_{kj}$$

where $\| z_{kj} \| < \eta/r$ for sufficiently large k.

From the definition of the integral and from Corollary 2.2 we obtain

$$\int_{\alpha_{p_j-1}}^{\alpha_{p_j}} DF(u(\tau), t) = \int_{\alpha_{p_j-1}}^{\tau_{p_j}} \ldots + \lim_{\varepsilon \to 0+} \int_{\tau_{p_j}}^{\tau_{p_j}+\varepsilon} \ldots + \lim_{\varepsilon \to 0+} \int_{\tau_{p_j}+\varepsilon}^{\alpha_{p_j}} \ldots =$$
$$= F(u(\tau_{p_j}), \tau_{p_j}+) - F(u(\tau_{p_j}), \tau_{p_j}) + z_j \quad (3.11)$$

where (see (3.5))

$$\| z_j \| \leqslant h(\alpha_{p_j}) - h(\tau_{p_j}+) + h(\tau_{p_j}) - h(\alpha_{p_j-1}) <$$
$$< h(\tau_{p_j} + \delta_j) - h(\tau_{p_j}+) + h(\tau_{p_j}) - h(\tau_{p_j} - \delta_j) < \eta/r$$

From (3.10) and (3.11) there results

$$\left\| \int_{\alpha_{p_j-1}}^{\alpha_{p_j}} DF_k(u_k(\tau), t) - \int_{\alpha_{p_j-1}}^{\alpha_{p_j}} DF(u(\tau), t) \right\| < \frac{2\eta}{r} \qquad (j = 1, \ldots, n) \quad (3.12)$$

for large enough k. From (3.9) and (3.12) we obtain

$$\left\| \int_{\sigma_1}^{\sigma_2} DF_k(u_k(\tau), t) - \int_{\sigma_1}^{\sigma_2} DF(u(\tau), t) \right\| \leqslant \eta\, [2 + 6(h(\sigma_2') - h(\sigma_1))]$$

Since η is any positive number, it follows that

$$\int_{\sigma_1}^{\sigma_3} DF_k(u_k(\tau), t) \to \int_{\sigma_1}^{\sigma_3} DF(u(\tau), t) \qquad \text{as } k \to \infty$$

$$u(\sigma_3) - u(\sigma_1) = \int_{\sigma_1}^{\sigma_3} DF(u(\tau), t) \qquad \text{for} \quad \sigma_3 \in \langle \sigma_1, \sigma_2' \rangle \cap H \tag{3.13}$$

$$u_k(\sigma_3) \to u(\sigma_3), \qquad u_k(\sigma_1) \to u(\sigma_1)$$

But the function $u(\sigma_3)$ and the integral in (3.13), considered as a function of the independent variable σ_3, represents a function continuous from the left and of bounded variation. Hence (3.13) is valid for $\sigma_3 \in \langle \sigma_1, \sigma_2' \rangle$. Therefore, $u(\tau)$ is a solution of equation (3.1), $u(\sigma_1) = x(\sigma_1)$, and because the solution $x(\tau)$ is unique, $u(\tau)$ for $\tau \in \langle \sigma_1, \sigma_2' \rangle$.

Since every convergent subsequence of the sequence $x_k(\tau)$ converges to $x(\tau)$, the sequence $x_k(\tau)$ converges to $x(\tau)$, and the Lemma 3.1 has thus been proved.

Passing to the proof of the theorem, we call attention to the following result. It follows from Theorem 2.1 and Corollary 2.1, and from the facts that $\sigma_1 \in H$ and

$$[h_k(t_2) - h_k(t_1)] \leqslant h(t_2) - h(t_1) \qquad \text{for } t_1, t_2 \in H,\ t_1 < t_2$$

that there exists a number $\sigma_1' > \sigma$ such that the assertion (c) of Theorem 3.1 is true on $\langle \sigma_1, \sigma_1' \rangle$.

Let σ_4 be the upper boundary of such numbers $\sigma_2' \in \langle \sigma_1, \sigma_2 \rangle$, for which the assertion (c) is true on $\langle \sigma_1, \sigma_2' \rangle$, and suppose $\sigma_4 < \sigma_2$. If $\sigma_4 \in H$, we take σ_2', $\sigma_5 \in H$, $\sigma_2' < \sigma_4 < \sigma_5$ sufficiently close to σ_4.

By the established lemma, $x_k(\sigma_2') \to x(\sigma_2')$, and since the distance of the point $(x(\sigma_4), \sigma_4)$ from the complement of the set G is a positive number, it follows from Theorem 2.1 and Corollary 2.1 that solution $x_k(\tau)$ (for large enough subscript) can be extended over the interval $\langle \sigma_2', \sigma_5 \rangle$ with the preservation of the validity of the assertion (c). If $\sigma_4 \bar{\in} H$, then for every $\epsilon > 0$ and for the point $(x(\sigma_4), \sigma_4) \in G_1$, by Theorem 3.1, one can find numbers δ_1, δ_2 occurring in the definition of unconditional convergence. Let the numbers σ_2', $\sigma_5 \in \langle \sigma_1, \sigma_2 \rangle$ satisfy the conditions

$$\sigma_4 - \delta_2 < \sigma_2' < \sigma_4 < \sigma_5 < \sigma_4 + \delta_2 \qquad (\sigma_2', \sigma_5 \in H)$$

Since $x(\sigma_2') \longrightarrow x(\sigma_2')$, the solutions $x_k(\tau)$ can be extended over the interval $\langle \sigma_2', \sigma_5 \rangle$ with the preservation of the condition (c) because of the definition of unconditional convergence: Thus we arrive at a contradiction and the theorem is proved.

The chief difficulty which one meets in trying to apply Theorem 3.1 is connected with the complicated concept of unconditional convergence. For this reason we give certain criteria which are sufficient to insure unconditional convergence.

Theorem 3.2. Let $h_k(t) \to h(t)$ uniformly on $\langle 0, T\rangle$ and $F(x, t)$ $F(x, t)$ uniformly in G. Then $F_k(x, t) \to F(x, t)$ and $G_1 = G_F$.

Remark 3.2. Under the conditions of Theorem 3.2, one can make use of Theorem 3.1 (retaining the assumption that $x(t)$ is a solution of equation (3.1). Since $x_k(\tau) \to x(\tau)$ for $\tau \in <\sigma_1, \sigma_2> \cap H$ and since $h_k(\tau) \to h(\tau)$ uniformly, it follows that $x_k(\tau) \to x(\tau)$ uniformly. In this case the requirement that $\sigma_1, \sigma_2 \in H$ is not necessary.

Proof. Let $(x_0, t_0) \in G_F$ and let ρ be a positive number not greater than the distance of either one of the points (x_0, t_0) and $(x_0 + F(x_0, t_0 +) - F(x_0, t_0), t_0)$ from the complement of G. Let $\epsilon > 0$ be given. We select positive numbers δ_1, δ_2 such that

$$\delta_1 < {}^1/_2\,\rho$$

$$\delta_2 + h(t_0+\delta_1) - h(t_0+) + h(t_0) - h(t_0-\delta_1) + \\ + \omega(\delta_2 + h(t_0) - h(t_0-\delta_1))\,[h(t_0+) - h(t_0)] < \min(\varepsilon, {}^1/_2\,\rho)$$

Let

$$\theta_k = \max(\sup_t |h_k(t) - h(t)|,\ \sup_{x,t} \|F_k(x, t) - F(x, t)\|)$$

and let $k > 0$ be such that

$$6\theta_k + \delta_2 + h(t_0+\delta_1) - h(t_0+) + h(t_0) - h(t_0-\delta_1) + \\ + \omega(2\theta_k + \delta_2 + h(t_0) - h(t_0-\delta_1))\,[h(t_0+) - h(t_0)] < \min(\varepsilon, {}^1/_2\,\rho) \quad (3.14)$$

$$t_0 - \delta_1 < t_1 < t_0 < t_2 < t_0 + \delta_1, \qquad \|y - x_0\| < \delta_2$$

According to Theorem 2.1 and Lemma 2.1 the solution $x_k(\tau)$ of equation (3.2), $x_k(t_1) = y$, exists on the interval $\langle t_1, t_0\rangle$, $k > K$ and we have

$$\|x_k(t_0) - x_0\| \leqslant \|x_k(t_1) - x_0\| + \|x_k(t_0) - x_k(t_1)\| < \delta_2 + h_k(t_0) - h_k(t_1) < \\ < \delta_2 + 2\theta_k + h(t_0) - h(t_0 - \delta_1)$$

From this it follows that

$$\|x_k(t_0) + F_k(x_k(t_0), t_0+) - F_k(x_k(t_0), t_0) - x_0 - F(x_0, t_0+) + F(x_0, t_0)\| < \\ < \delta_2 + 2\theta_k + h(t_0) - h(t_0-\delta_1) + 2\theta_k + \\ + \omega(\delta_2 + 2\theta_k + h(t_0) - h(t_0-\delta_1))\,[h(t_0+) - h(t_0)]$$

and the solution $x_k(\tau)$ can be extended over the interval $\langle t_0, t_2\rangle$. We thus obtain the final estimate:

$$\|x_k(t_2) - x_k(t_1) - F(x_0, t_0+) + F(x_0, t_0)\| \leqslant \\ \leqslant \|x_k(t_0) - x_k(t_1)\| + \|x_k(t_0+) - x_k(t_0) - F(x_0, t_0+) + F(x_0, t_0)\| + \\ + \|x_k(t_2) - x_k(t_0+)\| \leqslant$$

$$\leqslant h_k(t_0) - h_k(t_0 - \delta_1) + h_k(t_0 + \delta_1) - h_k(t_0 +) + 2\theta_k +$$
$$+ \| F(x_k(t_0), t_0 +) - F(x_k(t_0), t_0) - F(x_0, t_0 +) + F(x_0, t_0) \| <$$
$$< h(t_0) - h(t_0 - \delta_1) + h(t_0 + \delta_1) - h(t_0 +) + 6\theta_k +$$
$$+ \omega(\delta_2 + 2\theta_k + h(t_0) - h(t_0 - \delta_1))[h(t_0 +) - h(t_0)] < \varepsilon$$

The last of these inequalities follows from (3.14). This completes the proof of Theorem 3.2.

Since the uniform convergence of the functions $F_k(x,t)$ to $F(x,t)$ is a requirement in Theorem 3.2, the function $F(x,t)$ will be continuous if $F_x(x,t)$ are continuous functions. This means that Theorem 3.2 is not applicable for the investigation of the case when in the sequence of the classical equation there occur terms $d_k(t)$ which approximate to the so-called Dirac function

$$d_k(t) \geqslant 0, \qquad \int_{-1}^{t} d_k(\tau)\, d\tau \to 0 \ (\text{ or } \ 1) \text{ where } t < 0 \ (\text{ or } \ t > 0)$$

For this case we establish Theorem 3.3 and some corollaries to it.

Retaining the previous notation, we consider the subset consisting of points $(x,t) \in G$ such that

$$(y,t) \in G, \qquad \text{if } \ \| y - x \| \leqslant h(t+) - h(t)$$

Theorem 3.3. Let

$$\limsup_{k \to \infty} [h_k(t_2) - h_k(t_1)] \leqslant h(t_2) - h(t_1) \qquad (t_1 < t_2, \ t_1, t_2 \in H)$$
$$F_k(x,t) \to F(x,t) \qquad ((x,t) \in G_2, \ t \in H)$$

Suppose that for an arbitrary point $t_0 \tilde{\epsilon}\ H$)for which there exists an $x \in E_n$ such that $(x, t_0) \in G_2$) there exist linear subspaces $E^{(1)}$ and $E^{(2)}$ of the space E_n, and an increasing function $h_0(t)$ $(0 \leqslant t \leqslant T)$ continuous at the point t_0.

Let the following conditions be satisfied:

(g) the spaces $E^{(1)}$, $E^{(2)}$ are orthogonal; their intersection contains only the origin, and their algebraic sum is the entire space E_n. (In consequence of this requirement every vector $u \in E_n$ can be expressed in a unique way in the form

$$u = u^{(1)} + u^{(2)}, \qquad u^{(1)} \in E^{(1)}, \qquad u^{(2)} \in E^{(2)}$$

We will write

$F_k(x,t) = F_k^{(1)}(x,t) + F_k^{(2)}(x,t)$, $x(\tau) = x^{(1)}(\tau) + x^{(2)}(\tau)$, etc)

(h) $F(x, t_0 +) - F(x, t_0) = F(x + u, t_0 +) - F(x + u, t_0) \in E^{(2)}$
if $(x,t), (x+u, t) \in G$, $u \in E^{(2)}$

(i) $$\| F_k^{(2)}(x, t_2) - F_k^{(2)}(x, t_1) - F_k^{(2)}(x+u, t_2) + F_k^{(2)}(x+u, t_1) \| \leqslant \leqslant | h_0(t_2) - h_0(t_1) |$$

if $(x, t_2),\ (x, t_1),\ (x+u, t_2),\ (x+u, t_1) \in G,\ u \in E^{(2)}$.

(j) $$\| F_k^{(1)}(x, t_2) - F_k^{(1)}(x, t_1) \| \leqslant | h_0(t_2) - h_0(t_1) |$$

Under these conditions $F_k(x,t)$ converges unconditionally to $F(x,t)$ i.e.

$$F_k(x, t) \rightrightarrows F(x, t) \text{ in } G_2$$

Proof. Let $(x_0, t_0) \in G_2$, $t_0 \bar{\in} H$. On the basis of the hypotheses, and Theorem 2.1 and Corollary 2.1, one can easily deduce that there exist numbers $\tilde{\delta}_1 > 0$, $\tilde{\delta}_2 > 0$, $\tilde{K} > 0$ such that, if $t_0 - \tilde{\delta}_1 < t_1 < t_0$, $\|y - x_0\| < \tilde{\delta}_2$, $k > \tilde{K}$, one can determine the integral $x_k(t)$ of equation (3.2) on the interval $\langle t_1, t_0 + \tilde{\delta}_1 \rangle$ with $x_k(t_1) = y$.

Let $\epsilon > 0$ be given. For this ϵ and for the point (x_0, t_0) we determine $\delta_1 > 0$, $\delta_2 > 0$, $\delta_3 > 0$ such that the following inequalities are satisfied: $\delta_1 < \tilde{\delta}_1$, $\delta_2 < \tilde{\delta}_2$,

$$\begin{aligned} &\omega(\delta_2)[h(t_0+\delta_1) - h(t_0-\delta_1) + \delta_3] + \delta_3 + 2h_0(t+\delta_1) - 2h_0(t-\delta_1) + \\ &+ h(t_0+\delta_1) - h(t_0+) + h(t_0) - h(t_0-\delta_1) + \\ &+ \omega(h_0(t_0+\delta_1) - h_0(t_0-\delta_1))[h(t_0+\delta_1) - h(t_0-\delta_1) + \delta_3] < \varepsilon \end{aligned} \tag{3.15}$$

Let $t_0 - \delta_1 < t_1 < t_0 < t_2 < t_0 + \delta_1$, $t_1, t_2 \in H$, and let $K > \tilde{K}$ be such a large number that

$$\| F_k(x_0, t_2) - F(x_0, t_2) \| < {}^1/_2\,\delta_3, \quad \| F_k(x_0, t_1) - F(x_0, t_1) \| < {}^1/_2\,\delta_3$$

$$h_k(t_2) - h_k(t_1) < h(t_2) - h(t_1) + \delta_3 \text{ for } k > K$$

We next prove that the condition (III), which enters into the definition of unconditional convergence, is satisfied. The requirement (a) is satisfied because of the choice of the numbers $\tilde{\delta}_1$, $\tilde{\delta}_2$. Next, let $\|y - x_0\| < \delta_2$, $k > K$ and suppose that $x_k(\tau)$ is a solution of the equation (3.12) with $\tau \in \langle t_1, t_2 \rangle$ $x_k(t_1) = y$.

From requirement (j) and Corollary 2.1 it follows that

$$\begin{aligned} &\| x_k^{(1)}(t_3) - x_k^{(1)}(t_1) \| = \left\| \int_{t_1}^{t_3} DF_k^{(1)}(x_k(\tau), t) \right\| \leqslant \\ &\leqslant h_0(t_3) - h_0(t_1) < h_0(t_0+\delta_1) - h_0(t_0-\delta_1) \qquad (t_1 < t_3 \leqslant t_2) \end{aligned} \tag{3.16}$$

Since

$$\begin{aligned} &\| F_k^{(2)}(x_k^{(1)}(\tau) + x_k^{(2)}(\tau), t_5) - F_k^{(2)}(x_k^{(1)}(\tau) + x_k^{(2)}(\tau), t_4) - \\ &- F_k^{(2)}(x_k^{(1)}(t_1) + x_k^{(2)}(\tau), t_5) + F_k^{(2)}(x_k^{(1)}(t_1) + x_k^{(2)}(\tau), t_4) \| \leqslant \end{aligned}$$

$$\leqslant \omega(\|x_k^{(1)}(\tau) - x_k^{(1)}(t_1)\|)\,|h_k(t_5) - h_k(t_4)| <$$
$$< \omega(h_0(t_0 + \delta_1) - h_0(t_0 - \delta_1))\,|h_k(t_5) - h_k(t_4)|$$
$$(t_4,\ t_5, \tau \in \langle t_1, t_2 \rangle)$$

we have by Corollary 2.1

$$\left\| \int_{t_1}^{t_2} D\,[F_k^{(2)}(x_k^{(1)}(\tau) + x_k^{(2)}(\tau), t) - F_k^{(2)}(x_k^{(1)}(t_1) + x_k^{(2)}(\tau), t)] \right\| \leqslant$$
$$\leqslant \omega(h_0(t_0 + \delta_1) - h_0(t_0 - \delta_1))\,(h_k(t_2) - h_k(t_1))$$

Hence

$$x_k^{(2)}(t_2) - x_k^{(2)}(t_1) = \int_{t_1}^{t_2} DF_k^{(2)}(x_k^{(1)}(\tau) + x_k^{(2)}(\tau), t) =$$
$$= \int_{t_1}^{t_2} DF_k^{(2)}(x_k^{(1)}(t_1) + x_k^{(2)}(\tau), t) + z_k \tag{3.17}$$

where

$$\|z_k\| < \omega(h_0(t_0 + \delta_1) - h_0(t_0 - \delta_1))\,(h(t_2) - h(t_1) + \delta_3)$$

Furthermore, 'it follows from requirements (i) that

$$\|F_k^{(2)}(x_k^{(1)}(t_1) + x_k^{(2)}(\tau), t_5) - F_k^{(2)}(x_k^{(1)}(t_1) + x_k^{(2)}(t_1), t_5) -$$
$$- F_k(x_k^{(1)}(t_1) + x_k^{(2)}(\tau), t_4) + F_k(x_k^{(1)}(t_1) + x_k^{(2)}(t_1), t_4)\| \leqslant |h_0(t_5) - h_0(t_4)|$$

and by Corollary 2.1

$$\left\| \int_{t_1}^{t_2} D\,[F_k^{(2)}(x_k^{(1)}(t_1) + x_k^{(2)}(\tau), t) - F_k^{(2)}(x_k^{(1)}(t_1) + x_k^{(2)}(t_1), t)] \right\| \leqslant$$
$$\leqslant h_0(t_2) - h_0(t_1) < h_0(t_0 + \delta_1) - h_0(t_0 - \delta_1) \tag{3.18}$$

It is obvious that

$$\int_{t_1}^{t_2} DF_k^{(2)}(x_k^{(1)}(t_1) + x_k^{(2)}(t_1), t) = F_k^{(2)}(x_k(t_1), t_2) - F_k^{(2)}(x_h(t_1), t_1).$$

Using (3.17), we obtain

$$x_k^{(2)}(t_2) - x_k^{(2)}(t_1) = F_k^{(2)}(x_k(t_1), t_2) - F_k^{(2)}(x_k(t_1), t_1) + z_k + w_k \tag{3.19}$$

where $\|w_k\| < h_0(t_0 + \delta_1) - h_0(t_0 - \delta_1)$ and

$$x_k^{(2)}(t_2) - x_k^{(2)}(t_1) = F^{(2)}(x_0, t_2) - F^{(2)}(x_0, t_1) + s_k + z_k + w_k \tag{3.20}$$
$$\|s_k\| < \omega(\delta_2)\,[h(t_0 + \delta_1) - h(t_0 - \delta_1) + \delta_3]$$

Next, we have

$$F(x_0, t_0 +) - F(x_0, t_0) = F^{(2)}(x_0, t_0 +) - F^{(2)}(x_0, t_0) \in E^{(2)}$$

$$\begin{aligned}&\|F(x_0, t_0+) - F(x_0, t_0) - F_k^{(2)}(x_0, t_2) + F_k^{(2)}(x_0, t_1)\| = \\ &= \|F^{(2)}(x_0, t_0+) - F^{(2)}(x_0, t_0) - F_k^{(2)}(x_0, t_2) + F_k^{(2)}(x_0, t_1)\| \leqslant \\ &\leqslant \|F(x_0, t_2) - F(x_0, t_0+)\| + \|F(x_0, t_0) - F(x_0, t_1)\| + \\ &+ \|F_k(x_0, t_2) - F(x_0, t_2)\| + \|F_k(x_0, t_1) - F(x_0, t_1)\| \leqslant \\ &\leqslant h(t_0+\delta_1) - h(t_0+) + h(t_0) - h(t_1) + \delta_3\end{aligned} \tag{3.21}$$

From (3.16), (3.20) and (3.21) we obtain

$$\begin{aligned}&\|x_k(t_2) - x_k(t_1) - F(x_0, t_0+) + F(x_0, t_0)\| < \\ &< h_0(t_0+\delta_1) - h_0(t_0-\delta_1) + \|s_k\| + \|z_k\| + \|w_k\| + \\ &+ h(t_0+\delta_1) - h(t_0+) + h(t_0) - h(t_0-\delta_1) + \delta_3 < \varepsilon\end{aligned}$$

where the last inequality follows from (3.17), (3.19), (3.20) and (3.15). This completes the proof of the Theorem 3.3.

Let us now pass to the consideration of more concrete differential equations. Let

$$f_i(x_1, \ldots, x_n, t),\ g_i(x_1, \ldots, x_l) \qquad (i = 1, \ldots, n;\ 0 \leqslant l < n)$$

be real continuous functions of their arguments defined for $-\infty < x_i < \infty$, $-z \leqslant t \leqslant z, z > 1$ and $g_i(x_1, \ldots, x_l) \equiv 0$ for $i = 1, \ldots l$. (If $l = 0$, then all functions g_i reduce to constants).

Let $d_k(t)$ $(k = 3, 4, 5, \ldots)$ be continuous functions defined for $-z \leqslant t \leqslant z$ and satisfying the following conditions

$$\int_{-z}^{z} |d_k(t)|\, dt < C < \infty \tag{3.22}$$

$$D_k(t) = \int_{-z}^{t} d_k(\tau)\, d\tau \to 0 \ (\text{ or } 1), \qquad \text{if} \quad t < 0 \ (\text{ or } t > 0)$$

$$\int_{-z}^{-t} |d_k(\tau)|\, d\tau + \int_{t}^{z} |d_k(\tau)|\, d\tau \to 0 \ \text{ for } k \to \infty,\ 0 < t \leqslant z$$

We shall consider the system of differential equations

$$\frac{dx_i}{dt} = f_i(x_1, \ldots, x_n, t) + g_i(x, \ldots, x_l)\, d_k(t) \qquad (i = 1,\ldots,n;\ k = 3,4,5) \tag{3.23}$$

Let us set

$$F_i(x_1, \ldots, x_n, t) = \int_0^t f_i(x_1, \ldots, x_n, \tau)\, d\tau$$

$$B(t) = 0 \ (\text{ or } 1), \ \text{ if } \ -z \leqslant t \leqslant 0 \ (\text{ or } 0 < t \leqslant z)$$

and let us write the system (3.23) in the equivalent form (3.24)

$$\frac{dx_i}{d\tau} = D[F_i(x_1, \ldots, x_n, t) + g_i(x_1, \ldots, x_l)\, D_k(t)] \quad (i = 1,\ldots,n;\ k = 3,4,5,\ldots)$$

Let us consider the "limiting" system

$$\frac{dx_i}{d\tau} = D[F_i(x_1, \ldots, x_n, t) + g_i(x_1, \ldots, x_l)\, B(t)] \qquad (i = 1, \ldots, n) \tag{3.25}$$

Suppose that the solution of the system (3.25) is uniquely determined by the initial conditions. This is equivalent to assuring that the solution of the system

$$\frac{dx_i}{dt} = f_i(x_1, \ldots, x_n, t) \qquad (i = 1, \ldots, n) \tag{3.25*}$$

has the same property.

Let

$$(x_1(\tau), \ldots, x_n(\tau)) \qquad (-1 \leqslant \tau \leqslant +1)$$
$$\sup |x_j(\tau)| < M_1 \qquad (\tau \in \langle -1, +1 \rangle)$$

be the solution of the system (3.25). From the remark 0.1 and from (2.2*) is follows that $(x_1(t), \ldots, x_n(t))$ is a solution of the system (3.25*) on the interval $\langle -1, 0 \rangle$ and $\langle 0, 1 \rangle$ separately and that

$$\lim_{t \to 0+} x_i(t) = x_i(0) \qquad (i = 1, \ldots, l)$$
$$\lim_{t \to 0+} x_i(t) = x_i(0) + g_i(x_1(0), \ldots, x_l(0)) \qquad (i = l+1, \ldots, m)$$

We define the function $\tilde{g}_i(x_1 \ldots, x_l)$ in the following way:

$$\tilde{g}_i(x_1, \ldots, x_l) \equiv 0 \qquad (i = 1, \ldots, l)$$
$$\tilde{g}_{i'}(x_1, \ldots, x_l) = g_i(x_1, \ldots, x_l), \qquad \text{if } |x_j| \leqslant M_1 \quad (j = 1, \ldots, l)$$
$$\tilde{g}_i(x_1, \ldots, x_l) = g_i(y_1, \ldots, y_l), \qquad \text{if } \max_{j=1,\ldots,l} |x_j| > M_1$$
$$y_j = M_1 \frac{x_j}{\max\limits_{j=1,\ldots,l} |x_j|} \qquad (j = 1, \ldots, l;\ i = l+1, \ldots, n)$$

Obviously, $|g_i(x_1, \ldots, x_l)| < M_2$. Let us consider the systems

$$\frac{dx_i}{d\tau} = D\,[F_i(x_1, \ldots, x_n, t) + \tilde{g}_i(x_1, \ldots, x_l)\, D_k(t)] \tag{3.26}$$

$$\frac{dx_i}{d\tau} = D\,[F_i(x_1, \ldots, x_n, t) + \tilde{g}_i(x_1, \ldots, x_l)\, B(t)] \tag{3.27}$$

Since the systems (3.26) and (3.27) coincide on the set

$$|x_j| \leqslant M_1 \qquad (j = 1, \ldots, l;\ -\infty < x_l < \infty, \ldots, -\infty < x_n < \infty;\ -z < t < z)$$

with the systems (3.24) and (3.25), respectively, it follows that $x(\tau)$ is a solution of the equation (3.27). The set G is defined by the following inequalities

$$|x_j| < M_3, \qquad |t| < z$$

where $M_3 > M_1 + CM_2$ (see (3.22)). Obviously, $C \geqslant 1$. Let

$$M_4 = \max_{i=1,\ldots,n} f_i(x_1, \ldots, x_n, t) \quad \text{for} \quad (x_1, \ldots, x_n, t) \in \bar{G}$$

and let us set

$$h(t) = \sqrt{n}\,(M_4 t + CM_2 D(t)), \qquad h_k(t) = \sqrt{n}\Big(M_4 t + M_2 \int_{-z}^{t} |d_k(\tau)|\, d\tau\Big)$$

Since it is true that if a function is continuous on a compact set it is also uniformly continuous on that set, it follows that there exists a function $\omega(\eta)$, continuous on $0 \leqslant \eta \leqslant \eta_0$, with $\omega(0) = 0$ (η_0 is the diameter of the set G) such that

$$\left[\sum_{i=1}^{n} (f_i(\tilde{x}_1, \ldots, \tilde{x}_n, t) - f_i(x_1, \ldots, x_n, t))^2\right]^{1/2} \leqslant \omega\left(\left[\sum_{i=1}^{n} (\tilde{x}_i - x_i)^2\right]^{1/2}\right)$$

$$\left[\sum_{i=1}^{n} (g_i(\tilde{x}_1, \ldots, \tilde{x}_l) - g_i(x_1, \ldots, x_l))^2\right]^{1/2} \leqslant \omega\left(\left[\sum_{i=1}^{l} (\tilde{x}_i - x_i)^2\right)^{1/2}\right]$$

Let us write the equations (3.26) and (3.27) in the vector form:

$$\frac{dx}{dt} = DF_k(x, t), \qquad \frac{dx}{d\tau} = DF(x, t) \tag{3.28}$$

It can be verified that $\mathbf{F}(x,t) \in F(G, \omega, k)$, $F_k(x,t) \in \mathbf{F}(G, \omega, h_k)$ and that the set G_2 (see the definition of G_2 before Theorem 3.3) contains all the points,

$$(x_1, \ldots, x_n, 0), \text{ where } |x_j| < M_1 \qquad (j = 1, \ldots, n)$$

Let us prove that the conditions of Theorem (3.3) are satisfied. Obviously, the set H contains all the points t, $0 < t \leqslant z$,

$$\lim_{k\to\infty} \sup h_k(t_2) - h_k(t_1) \leqslant h(t_2) - h(t_1) \qquad (t_1 < t_2,\ t_1, t_2 \in H)$$

and $F_k(x,t) \to F(x,t)$ where $t \in H$ because of our assumptions about the sequence $d_k(t)$.

Let $E^{(1)}$ be the space of the points $(x_1, \ldots, x_l;\ 0, \ldots,)$ and $E^{(2)}$ be the space of points $(0, \ldots, 0;\ x_{l+1}, \ldots, x_n)$, and let us set $h_0 = 2\sqrt{n}M_4 t)$.

That the conditions (g) to (j) of the theorem are satisfied is obvious. The requirement (j) is fulfilled because the terms that contain the functions $\tilde{g}$ vanish. Hence, $F_k(x,t) \rightrightarrows F(x,t)$ in the region G_2.

Let y_k be a sequence of points in E_n, such that $y_k \to x(-1)$ as $k \to \infty$. According to Theorem 3.1, there exist solutions $x_k(\tau)$ of the equation (3.28), $x_k(-1) = y_k$ on the interval $\langle -1, +1 \rangle$ for all k sufficiently large, and $x_k(\tau) \to x(\tau)$ for $0 < |\tau| \leqslant 1$.

Since it is true that for $\eta > 0$

$$h_k(t_2) - h_k(t_1) \to h(t_2) - h(t_1) \qquad \text{as } k \to \infty$$

uniformly if $t_1, t_2 \in \langle \eta, 1 \rangle$, or if $t_1, t_2 \in \langle -1, -\eta \rangle$, it follows that $x_k(\tau) \to x(\tau)$ uniformly as $k \to \infty$ if $0 < \zeta < |\tau| < 1$.

Making use of the inequalities (3.16) we now obtain the result that $x_k^{(1)}(\tau) \to x^{(1)}(\tau)$ uniformly on the entire interval $\langle -1, +1 \rangle$. This means that the first coordinates $x_{ki}(\tau)$ of the functions $x_k(\tau)$ tend uniformly to the corresponding coordinates of the function $x(\tau)$ on the interval $\langle -1, +1 \rangle$. Taking into consideration the fact that

$|x_{ki}(\tau)| < M_3$ for k large enough, we conclude that the system of functions $(x_{k1}(\tau), \ldots, x_{kn}(\tau))$ is a solution of the equations (3.24) and (3.23). Thus the solution of the system (3.23) converges to the solution of the system (3.25), $t \neq 0$, and the first functions converge uniformly.

We call attention to the fact that all our considerations are applicable also to the sequence of equations of the form

$$\frac{d^{s+1}x}{dt^{s+1}} = f(x, \dot{x}, \ldots, x^s, t) + g\ (x, \dot{x}, \ldots, x^{s-1})\, d_k(t) \tag{3.29}$$

$x \in E_1$ (if $s = 0$, we take g = const), for the substitutions

$$x = x_1, \quad \frac{dx_1}{dt} = x_2, \ldots, \frac{dx_{s-1}}{dt} = x_s$$

will transform the equation (3.29) into the system (3.23). This means that the solutions $x_k(t)$, and their derivatives up to the order $s - 1$ will converge uniformly. In regard to the derivative of order s it can be said that this function will converge for $t \neq 0$ to a function which can be discontinuous. (If $s = 0$, then the solutions $x_k(t)$ converge only for $t \neq 0$.) It should be noted that x in equation (3.29) can be considered as an element in E_n.

4. Uniqueness of solutions

We stipulate that

$$\omega(\eta) = c\eta \quad \text{for} \quad \eta > 0, \quad c > 0.$$

Theorem 4.1. Let $F(x,t) \in \mathbf{F}(G, \omega, h)$, $(x_0, t_0) \in G_F$. Then, for any given interval $\langle t_0, t_0 + \sigma\rangle$, $\sigma > 0$ there exists at most one solution $x(t)$ of the equation

$$\frac{dx}{dt} = DF(x, t), \quad x(t_0) = x_0 \tag{4.1}$$

Remark 4.1. The uniqueness is not always preserved for the solutions $x(t)$ defined on the interval $\langle t_0 - \sigma, t_0\rangle$, $x(t_0) = x_0$ $(\sigma > 0)$ if such solutions exist. This follows from the example given in Remark 2.2.

The Theorem 4.1 is a direct consequence of the following proposition.

Let two solutions $x(\tau)$, $y(\tau)$ of equation (4.1) be given for $\epsilon < t_0$ $t_0 + \sigma >$. Then the following inequality is valid

$$\begin{aligned} \|x(\tau) - y(\tau)\| \leqslant \|x(t_0) - y(t_0)\| [1 + c(h(t_0+) - h(t_0))] \times \\ \times \exp\{c(h(\tau) - h(t_0+))\} \end{aligned} \tag{4.2}$$

In order to prove this inequality (4.2) we first establish two lemmas.

Lemma 4.1. Let the function $U(\tau, t)$ take on values j in some space E_n, and let the function $V(\tau, t)$ take on values in E_1, and let the integrals

$$\int_{\tau_1}^{\tau_2} DU(\tau, t), \quad \int_{\tau_1}^{\tau_2} DV(\tau, t)$$

exist. Suppose that

$$V(\tau, t) \leqslant V(\tau, \tau) \quad \text{for } t \leqslant \tau, \qquad V(\tau, t) \geqslant V(\tau, \tau) \quad \text{for } t \geqslant \tau$$

and suppose there exists a function $\delta(\tau) > 0$ $(\tau_1 \leqslant \tau \leqslant \tau_2)$ such that

$$\|U(\tau, t) - U(\tau, \tau)\| \leqslant |V(\tau, t) - V(\tau, \tau)|, \quad \text{if} \quad |t - \tau| < \delta(\tau), \ \tau_1 \leqslant \tau \leqslant \tau_2$$

Then

$$\left\| \int_{\tau_1}^{\tau_2} DU(\tau, t) \right\| \leqslant \int_{\tau_1}^{\tau_2} DV(\tau, t)$$

The proof of Lemma 4.1 can be carried out without difficulty on the basis of an equivalent concept of the integral as given in reference [1], Section 1, 1.2.

Lemma 4.2.

$$\int_{\tau_1}^{\tau_2} h^k(\tau)\, dh(\tau) \leqslant \frac{1}{k+1} [h^{k+1}(\tau_2) - h^{k+1}(\tau_1)] \qquad (k \geqslant 0)$$

We assume, as always, that $h(t)$ is an increasing function continuous from left and that

$$\int_{\tau_1}^{\tau_2} h^k(\tau)\, dh(\tau) = \int_{\tau_1}^{\tau_2} Dh^k(\tau)\, h(t)$$

(See [1], remark 1. 1. 2; the integral on the right side exists by Theorem 1.1, where $F(x,t) = xh(t)$, $u(\tau) = h^k(\tau)$).

Lemma 4.2 follows from the fact that for every $\epsilon > 0$ the function

$$\frac{1}{k+1} h^{k+1}(\tau) + \varepsilon h(\tau)$$

is an upper function for $h^k(\tau)h(t)$.

We shall now prove the inequality (4.2). Since $x(\tau)$ and $y(\tau)$ are functions of bounded variation, $\|x(\tau) - y(\tau)\| \leqslant K$, $\tau \in \langle t_0, t_0 + \sigma \rangle$.

Obviously,

$$x(\tau_2) - y(\tau_2) = x(t_0) + F(x(t_0), t_0+) - F(x(t_0), t_0) - y(t_0) -$$

$$- F(y(t_0), t_0+) + F(y(t_0), t_0) + \lim_{\tau_1 \to t_0+} \int_{\tau_1}^{\tau_2} D[F(x(\tau), t) - F(y(\tau), t)] \quad (4.3)$$

$$\|x(t_0) + F(x(t_0), t_0+) - F(x(t_0), t_0) - y(t_0) - F(y(t_0), t_0+) +$$

$$+ F(y(t_0), t_0)\| \leqslant \|x(t_0) - y(t_0)\| [1 + c(h(t_0+) - h(t_0))] = u$$

Since $\|x(\tau)\| < K$

$$\|[F(x(\tau), t) - F(y(\tau), t)] - [F(x(\tau), \tau) - F(y(\tau), \tau)]\| \leqslant cK\, |h(t) - h(\tau)|$$

Lemma 4.1 yields

$$\left\| \lim_{\tau_1 \to t_0+} \int_{\tau_1}^{\tau_2} D\,[F(x(\tau),\, t) - F(y(\tau),\, t)] \right\| \leqslant cK\,(h(\tau_2) - h(t_0+))$$

and from (4.3) we obtain the inequality

$$\| x(\tau_2) - y(\tau_2) \| \leqslant u + cK\,(h(\tau_2) - h(t_0+))$$

Let us assume that the following inequality already holds for some integer s.

$$\| x(\tau) - y(\tau) \| \leqslant u\left[1 + c\,(h(\tau) - h(t_0+)) + \ldots + \frac{\{c\,(h(\tau) - h(t_0+))\}^{s-1}}{(s-1)!}\right] + \\ + K\,\frac{\{c\,(h(\tau) - h(t_0+))\}^{s}}{s!} \tag{4.4}$$

From this it follows that

$$\| [F(x(\tau),\, t) - F(y(\tau),\, t)] - [F(x(\tau),\, \tau) - F(y(\tau),\, \tau)] \| \leqslant \\ \leqslant c\left\{u\left[1 + c\,(h(\tau) - h(t_0+)) + \cdots + \frac{\{c\,(h(\tau) - h(t_0+))\}^{s-1}}{(s-1)!} + \right.\right. \\ \left.\left. + K\,\frac{\{c\,(h(\tau) - h(t_0+))\}^{s}}{s!}\right]\right\} |h(t) - h(\tau)|$$

and by Lemmas 4.1 and 4.2 we have

$$\left\| \lim_{\tau_1 \to t_0+} \int_{\tau_1}^{\tau_2} D\,[F(x(\tau),\, t) - F(y(\tau),\, t)] \right\| \leqslant \\ \leqslant \lim_{\tau_1 \to t_0+} \int_{\tau_1}^{\tau_2} D\left[cn\left[1 + c\,(h(\tau) - h(t_0+)) + \ldots + \frac{\{c\,(h(\tau) - h(t_0+))\}^{s-1}}{(s-1)!}\right] + \right. \\ \left. + cK\,\frac{\{c\,(h(\tau) - h(t_0+))\}^{s}}{s!}\right](h(t) - h(t_0+)) = \\ = n\left[c\,(h(\tau_2) - h(t_0+)) + \ldots + \frac{\{c\,(h(\tau_2) - h(t_0+))\}^{s}}{s!}\right] + K\,\frac{\{c\,(h(\tau_2) - h(t_0+))\}^{s+1}}{(s+1)!}$$

From (4.3) it now follows that the inequality (4.4) is valid for $s + 1$. Since it holds for $s = 1$, it is valid for all integers n. Taking the limit, we obtain (4.2).

We call attention to the fact, that one can prove in an analogous way that the successive approximations of Picard will converge under our assumptions.

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Translated by H.P.T.